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MULTIPLE INTEGRALS INVOLVING PRODUCT
OF MODIFIED BESSEL FUNCTIONS OF THE
SECOND KIND

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ABSTRACT

The following two multiple integrals are evaluated

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k_r-1} K_{\nu_r}(\lambda_r) d\lambda_r \, K_\mu \left[x(\lambda_1 \ldots \lambda_{m-1})^{\pm 1} \right] \quad .$$

Also the asymptotic behavior of the two integrals as $|x| \to 0$ and as $|x| \to \infty$ is given.

MULTIPLE INTEGRALS INVOLVING PRODUCT
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F. M. Ragab

1. <u>Introductory.</u> It is proposed to establish the two following integrals:

If $m = 2, 3, 4, \ldots,$

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k_r-1} K_{\nu_r}(\lambda_r)\, d\lambda_r\, K_\mu\left(\frac{x}{\lambda_1 \lambda_2 \cdots \lambda_{m-1}}\right)$$

$$\sum_{r=1}^{m-1} k_r + 1 - 2m$$
$$= 2$$

$$\times \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left[\frac{\nu_1 + k_1}{2}, \frac{k_1 - \nu_1}{2}, \ldots, \frac{k_{m-1} + \nu_{m-1}}{2}, \frac{k_{m-1} - \nu_{m-1}}{2}, \frac{\mu}{2}, \frac{-\mu}{2}, 1 :: \frac{e^{i\pi}}{2^{2m}} x^2\right]$$

$$, \ldots \quad (1)$$

where x is real and positive and the symbol $\sum_{i,-i}$ means that in the expression

following it i is to be replaced by $-i$ and the two expressions are to be added.

If $R(k_r \pm \nu_r \pm \mu) > 0,$ $r = 1, 2, \ldots, m-1$,

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k_r-1} K_{\nu_r}(\lambda_r)\, d\lambda_r\, K_\mu(x \lambda_1 \lambda_2 \cdots \lambda_{m-1})$$

$$= 2^{\sum_{r=1}^{k} k_r - 2m + 1} \quad \pi \cdot \sum_{\mu,-\mu} \frac{1}{\sin \mu\pi} \left(2^{2m-4} x^2\right)^{-\frac{\mu}{2}}$$

$$\times E\left[\frac{k_1 + \nu_1 - \mu}{2}, \frac{k_1 - \nu_1 - \mu}{2}, \ldots, \frac{k_{m-1} + \nu_{m-1} - \mu}{2}, \frac{k_{m-1} - \nu_{m-1} - \mu}{2} : \frac{e^{\pm i\pi}}{2^{2m-4} x^2}\right],$$
$$1 - \mu$$

$$(2)$$

where $m = 2, 3, 4, \ldots,$ x is real and positive and the symbol $\sum\limits_{\mu, -\mu}$ has the same meaning as before. The function appearing in (1) and (2) is Mac Robert's E-function, the definitions and properties of which are to be found in [1]; pp. 348-352 and in [2]; pp. 203-206. The following formulae will be required in the proofs:

If $R(s \pm \mu) > 0,$ then ([3], p. 197 and p. 7)

$$\int_0^\infty \lambda^{s-1} K_\mu(\lambda) \, d\lambda = 2^{s-2} \, \Gamma(\frac{s+\mu}{2}) \, \Gamma(\frac{s-\mu}{2}), \tag{3}$$

$$K_\mu(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s-2} \, \Gamma(\frac{s+\mu}{2}) \, \Gamma(\frac{s-\mu}{2}) \, x^{-s} \, ds \tag{4}$$

If $P \geq q + 1,$ then ([1], p. 409)

$$E(p; \alpha_r : q; \rho_s : z) = \sum_{r=1}^p \prod_{s=1}^p{}' \Gamma(\alpha_s - \alpha_r) \{\Gamma(\rho_t - \alpha_r)\}^{-1} \Gamma(\alpha_r) z^{\alpha_r}$$

$$\times F \left[\begin{array}{c} \alpha_r, \, \alpha_r - \rho_1 + 1, \, \ldots, \, \alpha_r - \rho_q + 1 ; (-1)^{p-q} z \\ \alpha_r - \alpha_1 + 1, \, \ldots * \ldots, \, \alpha_r - \alpha_p + 1 \end{array} \right] \tag{5}$$

where $|\arg z| < \frac{1}{2}(p - q + 1)\pi;$ the prime in the product sign signifies that the factor for which $s = r$ is left and the asterisk in the F function means that the parameter $\alpha_r - \alpha_r + 1$ is omitted.

If $p \leq q,$ then ([1], p. 352)

$$E(p; \alpha_r : q, \rho_s : z) = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\rho_1) \ldots \Gamma(\rho_q)} \, F \left(\begin{array}{c} \alpha_1, \, \ldots, \, \alpha_p ; -\frac{1}{z} \\ \rho_1, \, \ldots, \, \rho_q \end{array} \right) . \tag{6}$$

If m is a positive integer

$$\Gamma(z)\ \Gamma(z+\tfrac{1}{m})\ \cdots\ \Gamma(z+\tfrac{m-1}{m}) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}}\ m^{\frac{1}{2}-mz}\ \Gamma(mz)\ . \tag{7}$$

If m is a positive integer, then ([4], p. 759)

$$\sum_{i,-i}\frac{1}{i}\,E(p;\ m\alpha_r\!:\!q;\ m\rho_s\!:\!z\,e^{i\pi}) =$$

$$(2\pi)^{-\frac{1}{2}(m-1)(p-q-1)}\ m^{m(\Sigma\alpha_r-\Sigma\rho_s)-\frac{1}{2}(p-q-1)}$$

$$\times \sum_{i,-i}\frac{1}{i}\,E\left\{\begin{array}{l}\alpha_1,\ \alpha_1+\frac{1}{m},\ldots,\alpha_1+\frac{m-1}{m};\ldots;\alpha_p,\ \alpha_p+\frac{1}{m},\ldots,\alpha_p+\frac{m-1}{m}:\ (\frac{z}{m^{p-q-1}})^m\,e^{i\pi} \\[2mm] \frac{1}{m},\ \frac{2}{m},\ldots,\frac{m-1}{m},\ \rho_1,\ \rho_1+\frac{1}{m},\ldots,\rho_1+\frac{m-1}{m};\ldots;\rho_q,\ \rho_q+\frac{1}{m},\ldots,\frac{\rho_q+m-1}{m}\end{array}\right\}$$

where $p > q+1$ and $|\arg z| < \frac{1}{2}(p-q-1)\pi$. $\tag{8}$

If $|\arg z| < \frac{1}{2}(p-q+1)\pi$, then, ([1], p. 374)

$$E(p;\alpha_r\!:\!q;\rho_s\!:\!z) = \frac{-1}{2\pi i}\int\Gamma(\xi)\frac{\prod\limits_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod\limits_{t=1}^{q}\Gamma(\rho_t-\xi)}\,z^{\xi}\,d\xi \tag{9}$$

where the integral is taken up the μ-axis, with loops, if necessary, to ensure that the pole at the origin lies on the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_\rho$ to the right of the contour. The contour must be modified if $p < v+1$ and $p = q+1$, $|z| < 1$.

Also the proof depends on the expression in terms of E-function of the generalized E-function

$$E\begin{pmatrix} p; \alpha_r & m; \rho_{q+s} & : z \\ & \Big/ & \\ q; \rho_s & \alpha_{p+r} & \end{pmatrix} =$$

$$\frac{1}{2\pi i} \int \Gamma(\xi) \frac{\displaystyle\prod_{r=1}^{p} \Gamma(\alpha_r - \xi) \prod_{s=1}^{m} \Gamma(\xi - \rho_{q+s} + 1)}{\displaystyle\prod_{s=1}^{v} \Gamma(\rho_s - \xi) \prod_{r=1}^{\ell} \Gamma(\xi - \alpha_{p+r} + 1)} z^{\xi} \, d\xi \quad , \tag{10}$$

where ℓ and m are positive integers and the contour passes up the η-axis

from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of the in-

tegrand at the origin and at $\rho_{q+1}{}^{-1}, \ \rho_{q+2}{}^{-1}, \ \cdots, \ \rho_{q+m}{}^{-\lambda}$ lie to the left and the

poles at $\alpha_1, \ \alpha_2, \ \cdots, \ \alpha_p$ to the right of the contour. When necessary the con-

tour is bent to the left or to the right at both ends until it is parallel to the

ξ-axis.

This expansion ([1], p. 419) is

$$E\begin{pmatrix} p; \alpha_r & m; \rho_{q+s} & : z \\ & \Big/ & \\ q; \rho_s & \ell; \alpha_{p+r} & \end{pmatrix} = \pi^{m-\ell} \prod_{r=1}^{\ell} \sin(\alpha_{p+r}\, \pi)$$

$$\prod_{s=1}^{m} \operatorname{cosec}(\rho_{q+s}\, \pi) \, E(p+\ell; \alpha_r : q+m; \rho_s : wz)$$

$$-\sum_{s=1}^{m} \pi^{m-\rho} \frac{\displaystyle\prod_{r=1}^{\ell} \sin(\rho_{q+s} - \alpha_{p+r})\,\pi}{\sin(\rho_{q+s}\,\pi) \displaystyle\prod_{t=1}^{m}{}' \sin(\rho_{q+s} - \rho_{q+t})\,\pi} z^{\rho_{q+s}-1}$$

$$\times E\begin{pmatrix} p+\ell; \ \alpha_r - \rho_{q+s} + 1 & & : wz \\ & & \\ 2 - \rho_{q+s}, \ \rho_1 - \rho_{q+s} + 1, \ \ldots *\ldots, \ \rho_{q+m} - \rho_{q+s} + 1 \end{pmatrix} \tag{11}$$

where the prime and the asterisk denote that the factor $\sin(\rho_{q+s} - \rho_{q+s})\pi$ and

the parameter $\rho_{q+s} - \rho_{q+s} + 1$ are omitted and ω is equal to 1 or $e^{\pm i\pi}$ according

as $\ell + m$ is even or odd.

To familiarize ourselves with the E-function the following relations may be

worth noting; they will, however, not be used here. From the definitions (5)

and (6) of the E-function it is clear that the E-function is immediately related

to the generalized hypergeometric function and reduces to simple expressions in

the ordinary or Gauss hypergeometric function when $p = 2$, $q = 1$. For $p = 1$

and $q = 1$ it is also evident that the E-function reduces to the confluent hyper-

geometric function or Kummer's function. The case $p = 2$, $q = 0$ yields the

relations [see [1], p. 351].

$$\cos n\pi \, E(\tfrac{1}{2} + n, \, \tfrac{1}{2} - n :: 2z) = \sqrt{(2\pi z)} \; e^z K_n(z), \tag{12}$$

$$E(\tfrac{1}{2} - k + m, \, \tfrac{1}{2} - k - m :: z) = \Gamma(\tfrac{1}{2} - k - m) \; \Gamma(\tfrac{1}{2} - k + m) \; z^{-k} e^{\frac{1}{2} z} \; W_{k, m}(z) \tag{13}$$

where $K_v(z)$ and $W_{k, m}(z)$ are the modified Bessel function of the second kind

and Whittaker function. Also it is immediate from the definition of the E-function

that for $p = q = 0$ the E-function is just $e^{-1/z}$ and we have

$$E(:: z) = e^{-\frac{1}{z}} . \tag{14}$$

More complicated parameters in the E-functions lead to the equivalence of the

E-function with products of Hankel functions, with Lommel functions and with

products of Whittaker functions. Some examples of this are:

$$K(z) = \frac{1}{4\pi} \sum_{i, -i} \frac{1}{i} \, E(1, \, \tfrac{1}{2}\mu, \, -\tfrac{1}{2}\mu :: \tfrac{1}{4} x^2 \, e^{i\pi}), \tag{15}$$

$$x^{\mu} \, H_{\nu}^{(1)}(x) \, H_{\nu}^{(2)}(x) = \pi^{-\frac{5}{2}} \, 2 \cos \nu\pi \, x^{\mu-1} \, E(\nu + \tfrac{1}{2} - \nu, \tfrac{1}{2} :: x^2)$$ (16)

$$S_{\mu, \nu}(x) = \{ \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\nu) \, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu) \}^{-1} (\tfrac{x}{2})^{\mu-1}$$

$$\times E(1, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu, \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\nu :: \tfrac{1}{4} x^2)$$ (17)

$$W_{k, m}(x) \, W_{-k, m}(x) = \frac{1}{2\pi^{3/2}}$$

$$\times \sum_{i, -i} \frac{1}{i} \, E \begin{bmatrix} 1, 1, \tfrac{1}{2}, \tfrac{1}{2} + m, \tfrac{1}{2} - m : \tfrac{1}{4} x^2 \, e^{i\pi} \\ 1 + k, \, 1 - k \end{bmatrix}$$ (18)

$$W_{k, m}(2ix) \, W_{k, m}(-2ix) = \pi^{-\frac{1}{2}} (\tfrac{x}{2})^{2k}$$

$$\times \{ \Gamma(\tfrac{1}{2} - k + m) \, \Gamma(\tfrac{1}{2} - k - m) \}^{-1} E \begin{bmatrix} \tfrac{1}{2} - k + m, \tfrac{1}{2} - k - m, \tfrac{1}{2} - k, 1 - k : \tfrac{x^2}{4} \\ 1 - 2k \end{bmatrix}$$

$$x^{2\mu} \, K_{2\nu}\left(xe^{\frac{\pi i}{4}}\right) K_{2\nu}\left(xe^{-\frac{\pi i}{4}}\right) = 2^{3\mu - 3} \pi^{-\frac{1}{2}} \, ,$$ (19)

$$\times \frac{1}{2\pi} \sum_{i, -i} \frac{1}{i} \, E\left(\tfrac{1}{2}\mu + \nu, \tfrac{1}{2}\mu - \nu, \tfrac{1}{2}\mu, \tfrac{1}{2}\mu + \tfrac{1}{2}, 1 :: \tfrac{x^4}{64} e^{i\pi} \right) \, .$$ (20)

§2. Proofs of the integrals.

To prove (1), apply (4) to replace $K_{\mu}\left(\dfrac{x}{\lambda_1 \dots \lambda_{m-1}} \right)$ on the left of (1) by

$$\frac{1}{2\pi i} \int 2^{s-2} \, \Gamma\left(\frac{s}{2} + \frac{\mu}{2} \right) \, \Gamma\left(\frac{s}{2} - \frac{\mu}{2} \right) \left(\frac{x}{\lambda_1 \dots \lambda_{m-1}} \right)^{-s} ds \, .$$

Then, on changing the order of integration and evaluating integrals by means of (3), the multiple integral becomes

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$$\frac{1}{2\pi i} \int 2^{s-2} \Gamma(\frac{s}{2}+\frac{\mu}{2}) \; \Gamma(\frac{s}{2}-\frac{\mu}{2}) \; x^{-s}$$

$$\times \prod_{r=1}^{m-1} \{2^{k_r+s-2} \; \Gamma(\frac{s}{2}+\frac{k_r}{2}+\frac{\nu_r}{2}) \; \Gamma(\frac{s}{2}+\frac{k_r}{2}-\frac{\nu_r}{2})\} \, ds = 2^{\sum_{r=1}^{m-1} k_r - 2m+1} \times$$

$$\frac{1}{2\pi i} \int \Gamma(\frac{\mu}{2}-\xi) \; \Gamma(\frac{-\mu}{2}-\xi) \prod_{r=1}^{m-1} \{\Gamma(\frac{k_r}{2}+\frac{\nu_r}{2}-\xi) \; \Gamma(\frac{k_r}{2}-\frac{\nu_r}{2}-\xi)\} (\frac{z^2}{4^m})^\xi \, d\xi$$

$$= 2^{\sum_{r=1}^{m-1} k_r - 2m+1} \times \frac{1}{2\pi i} \int \Gamma(\frac{\mu}{2}-\xi) \; \Gamma(-\frac{\mu}{2}-\xi)$$

$$\prod_{r=1}^{m-1} \{\Gamma(\frac{k_r}{2}+\frac{\nu_r}{2}-\xi) \; \Gamma(\frac{k_r}{2}-\frac{\nu_r}{2}-\xi)\} (\frac{z^2}{4^m})^\xi \{\frac{e^{i\pi\xi}-e^{-i\pi\xi}}{2\pi i}\} \Gamma(\xi) \; \Gamma(1-\xi) \, d\xi \; ,$$

where we have used the relation

$$\Gamma(\xi) \; \Gamma(1-\xi) = \frac{\pi}{\sin \pi\xi} \; .$$

Now apply (9) with $p = 2m+1$, $q = 0$ and so obtain (1) .

Again on substituting for $K_\mu(x\lambda_1 \ldots \lambda_{m-1})$ on the left of (2) from (4) and changing the order of integration as before, the multiple integral is found to be equal to

$$\frac{1}{2\pi i} \int 2^{s-2} \Gamma(\frac{s}{2}+\frac{\mu}{2}) \; \Gamma(\frac{s}{2}-\frac{\mu}{2}) \prod_{r=1}^{m-1} \{2^{k_r-s-2} \; \Gamma(\frac{k_r}{2}+\frac{\nu_r}{2}-\frac{s}{2})$$

$$\Gamma(\frac{k_r}{2}-\frac{\nu_r}{2}-\frac{s}{2})\} z^{-s} \, ds$$

$$= 2^{\sum_{r=1}^{m-1} k_r - 2m+1} \times \frac{1}{2\pi i} \int \Gamma(\frac{\mu}{2}+\xi) \; \Gamma(-\frac{\mu}{2}+\xi)$$

$$\times \prod_{r=1}^{m-1} \{\Gamma(\frac{k_r}{2}+\frac{\nu_r}{2}-\xi) \; \Gamma(\frac{k_r}{2}-\frac{\nu_r}{2}-\xi)\} (\frac{z^{-2}}{2^{2m-4}})^\xi \, d\xi \; ,$$

and on applying (11) with $m = 2$, $q = \ell = 0$, the expression on the right of (2) is obtained.

§3. Particular cases.

In (1) take $k_r = \dfrac{2r}{m}$, $\nu_r = \mu$ $(r = 1, 2, \ldots, m-1)$ and it becomes

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{\frac{2r}{m}-1} K_\mu(\lambda_r)\, d\lambda_r\, K_\mu\left(\frac{x}{\lambda_1 \cdots \lambda_{m-1}}\right) =$$

$$= 2^{-m} \times \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i} E\left(\frac{\mu}{2},\ \frac{\mu}{2}+\frac{1}{m},\ \ldots,\ \frac{\mu}{2}+\frac{m-1}{m},\ -\frac{\mu}{2},\right.$$

$$\left.-\frac{\mu}{2}+\frac{1}{m},\ \ldots,\ -\frac{\mu}{2}+\frac{m-1}{m},\ 1 :: e^{i\pi}\frac{x^2}{2^{2m}}\right)\ .$$

Now apply (8) with $p = 3$, $q = 0$, $\alpha_1 = \dfrac{1}{m}$, $\alpha_2 = \dfrac{\mu}{2}$, $\alpha_3 = -\dfrac{\mu}{2}$, then the last expression becomes

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{\frac{2r}{m}-1} K_\mu(\lambda_r)\, d\lambda_r\, K_\mu\left(\frac{x}{\lambda_1 \cdots \lambda_{m-1}}\right) = \pi^{m-1}$$

$$\frac{1}{4\pi} \sum_{i,-i} \frac{1}{i} E\left(1,\ \frac{m}{2}\mu,\ -\frac{m}{2}\mu :: e^{i\pi}mx^{\frac{1}{m}}\right) = \pi^{m-1} K_{m\mu}\left(mx^{\frac{1}{m}}\right), \quad \text{by (15)}\ .$$

Thus we have, if $x > 0$

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{\frac{2r}{m}-1} K_\mu(\lambda_r)\, d\lambda_r\, K_\mu\left(\frac{x}{\lambda_1 \cdots \lambda_{m-1}}\right) = \pi^{m-1} K_{m\mu}\left(mx^{\frac{1}{m}}\right)\ . \qquad (21)$$

Formula (21) is a generalization of Hardy's formula $(m = 2)$ ([5], p. 190) namely

$$\int_0^\infty K_\mu(\lambda)\, K_\mu\left(\frac{x}{\lambda}\right) d\lambda = \pi K_{2\mu}(2\nu(x))\ , \qquad (22)$$

where $x > 0$.

In (1) take $k_r = \dfrac{r}{m}$, $\mu = \nu_r = \pm\dfrac{1}{2}$ $(r = 1, 2, \ldots, m-1)$, apply the formula

$$K_{\pm\frac{1}{2}}(x) = \sqrt{\left(\frac{\pi}{2x}\right)}\, e^{-x} , \qquad\qquad (23)$$

and it becomes

$$\prod_{r=1}^{m-1} \int_0^\infty e^{-\lambda_r} \lambda_r^{\frac{r}{m}-1}\, d\lambda_r\, e^{-\frac{x}{\lambda_1 \ldots \lambda_{m-1}}} = 2^{\frac{1}{2}-m}\, \pi^{-\frac{1}{2}m}\, x^{\frac{1}{2}}$$

$$\times \frac{1}{2\pi} \sum_{i,-i} \frac{1}{i}\, E(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}+\frac{1}{2m}, -\frac{1}{4}+\frac{1}{2}m, \ldots$$

$$\ldots, \frac{1}{4}+\frac{m-1}{m}, -\frac{1}{4}+\frac{m-1}{m}, 1 \, \vdots\vdots\, e^{i\pi}\frac{x^2}{2m})$$

$$= x^{\frac{1}{2}}\frac{1}{2\pi} \sum_{i,-i} \frac{1}{i}\, E(-\frac{1}{2}, -\frac{1}{2}+\frac{1}{m}, \ldots, -\frac{1}{2}+\frac{m-1}{m}, 1 \, \vdots\vdots\, e^{i\pi} x)$$

$$= x\frac{1}{2\pi} \sum_{i,-i} \frac{1}{i}\, E(-1, -1+\frac{i}{m}, \ldots, -1+\frac{m-1}{m}, 1 \, \vdots\vdots\, e^{i\pi} x)$$

$$= x\,(2\pi)^{\frac{1}{2}(m-1)}\, m^{m-\frac{1}{2}}\, E(-m, 1 \, \vdots\vdots\, e^{i\pi} x^{\frac{1}{m}} m)$$

by (8) with $\alpha_1 = -1$, $\alpha_2 = \dfrac{1}{m}$, $q = 0$, and so we obtain

$$\int_0^\infty e^{-\lambda_1} \lambda_1^{\frac{1}{m}-1}\, d\lambda_1 \ldots \int_0^\infty e^{-\lambda_{m-1}} \lambda_{m-1}^{\frac{m-1}{m}-1}\, d\lambda_{m-1}\, e^{-\frac{x}{\lambda_1 \ldots \lambda_{m-1}}}$$

$$= (2\pi)^{\frac{1}{2}(m-1)}\, m^{-\frac{1}{2}}\, e^{-mx^{\frac{1}{m}}} \qquad\qquad (24)$$

where x is real and positive.

Again in (1), take $m = 2$, and expand each E-function on the right of (1) by means of (5) and combine the two resulting expressions, so getting,

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) \, K_\mu(\tfrac{x}{\lambda}) \, d\lambda = \sum_{\mu,-\mu} 2^{k+2\mu-1} \, \Gamma(\mu) \, \Gamma(\tfrac{k+\mu+\nu}{2}) \, \Gamma(\tfrac{k+\mu-\nu}{2})$$

$$\times \, {}_0F_3(;1 \;\; \mu, \, 1-\tfrac{k+\nu+\mu}{2}, \, 1-\tfrac{k+\mu-\nu}{2}; \, \tfrac{x^2}{16})$$

$$+ \sum_{\nu,-\nu} 2^{-k-2\nu-3} \, \Gamma(-\nu) \, \Gamma(\tfrac{-k+\mu-\nu}{2}) \, \Gamma(\tfrac{-k-\mu-\nu}{2}) \, x^{k+\nu}$$

$$\times \, {}_0F_3(;1+\nu, \, 1+\tfrac{k-\nu+\mu}{2}, \, 1+\tfrac{k+\mu+\nu}{2} \; ; \, \tfrac{x^2}{16}) \tag{25}$$

where $x > \sigma$ and each of the symbols $\displaystyle\sum_{\mu,-\mu}$, $\displaystyle\sum_{\nu,-\nu}$ has the same

meaning as before.

In (2), take $m = 2$, and get

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) \, K_\mu(x\lambda) \, d\lambda = 2^{k-3}$$

$$\sum_{\mu,-\mu} x^{-\mu} \, \Gamma(\tfrac{k+\nu-\mu}{2}) \, \Gamma(\tfrac{k-\nu-\mu}{2}) \, \Gamma(\mu)$$

$$_2F_1(\tfrac{k+\nu-\mu}{2}), \, \tfrac{k-\nu-\mu}{2}, \, 1-\mu \, ; \, x^2) \; .$$

Now sum the two hypergeometric series of Gauss by the formula

$$F(a,b;c;2) = \sum_{a,b} (1-z)^{-a} \frac{\Gamma(c) \, \Gamma(b-a)}{\Gamma(b) \, \Gamma(c-a)} \, {}_2F_1(a, \, c-b, \, a-b+1; \, \tfrac{1}{1-2})$$

$$\text{where} \; |\arg(1-z)| < \pi, \tag{25}$$

and so obtain Titchmarch's formula ([6] p. 98) namely

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(x\lambda) d\lambda = 2^{k-3} \{\Gamma(k)\}^{-1} x^{-\nu-k}$$

$$\Gamma(\tfrac{k+\mu+\nu}{2}) \; \Gamma(\tfrac{k+\mu-\nu}{2}) \; \Gamma(\tfrac{k+\nu-\mu}{2}) \; \Gamma(\tfrac{k-\mu-\nu}{2})$$

$$_2F_1(\tfrac{k+\mu+\nu}{2}, \tfrac{k+\nu-\mu}{2}; k; 1-\tfrac{1}{x^2}) \; , \tag{26}$$

where $R(k \pm \mu \pm \nu) > 0$ and x is real and positive.

In (2) take $m = 3$, and obtain the double integral

$$\int_0^\infty \int_0^\infty \lambda_1^{k_1-1} K_{\nu_1}(\lambda_1) \lambda_2^{k_2-1} K_{\nu_2}(\lambda_2) K_\mu(x\lambda_1\lambda_2) d\lambda_1 d\lambda_2$$

$$= 2^{k_1 + k_2 - 5} \pi \sum_{\mu, -\mu} \left\{ \cosec(\mu\pi) \; (16x^2)^{-\frac{\mu}{2}} \right.$$

$$\times E\left[\begin{matrix} \frac{k_1+\nu_1-\mu}{2}, \; \frac{k_1-\nu_1-\mu}{2}, \; \frac{k_2+\nu_2-\mu}{2}, \; \frac{k_2-\nu_2-\mu}{2} : \frac{e^{\pm i\pi}}{4x2} \\ 1 - \mu \end{matrix}\right] \left.\right\} , \tag{27}$$

where

$$R(k_1 \pm \nu_1 \pm \mu) > 0, \quad R(k_2 \pm \nu_2 \pm \mu) > 0, \; x \text{ is real and positive.}$$

§3. Asymptotic behavior of the integrals:

Now we are in a position to find the asymptotic expansion of the two multiple integrals (1) and (2) for large and small values of $|x|$; for the asymptotic expansion of the E-function is given by Mac Robert ([1], p. 358) .

If $p > q + 1$

$E(p; \alpha_r: q; \rho_s : z) \sim$ the first m terms of the divergent series

$$\frac{\Gamma(\alpha_1) \; \cdots \; \Gamma(\alpha_p)}{\Gamma(\rho_1) \cdots \; \Gamma(\rho_q)} \; _pF_\nu (p; \alpha_r; q; \rho_s; -\tfrac{1}{z}) + R_{m+1} \tag{28}$$

where R_{m+1} is a remainder which tends to zero as $|z| \to \infty$ and $m \to \infty$.

And the other functions that occur can be expressed in terms of ordinary generalized hypergeometric functions whose asymptotic expansions have been investigated by several writers (Barnes, [7] and [8], Wright [9]). However, an alternative proof for the asymptotic expansion of the integral (1) as $|x| \to \infty$ will be established. Thus we shall prove, for large values of $|x|$, the following asymptotic expansions

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k_r-1} K_{\nu_r}(\lambda_r) \, d\lambda_r \, K_\mu \left(\frac{x}{\lambda_1 \cdots \lambda_{m-1}} \right) \sim$$

$$2^{\sum_{r=1}^{m-1} k_r - 2m+1} \exp\left\{ -2m \left(\frac{x^2}{2^{2m}} \right)^{\frac{1}{2m}} \right\} \left(\frac{x^2}{2^{2m}} \right)^{\frac{1}{2m}\left[\sum_{r=1}^{m-1} k_r + \frac{1}{2} - m \right]}$$

$$\times \left\{ \frac{(2\pi)^{m-\frac{1}{2}}}{\sqrt{(2m)}} + \frac{M_1}{\left(\frac{x^2}{2^{2m}} \right)^{\frac{1}{2m}}} + \frac{M_2}{\left(\frac{x^2}{2^{2m}} \right)^{\frac{2}{2m}}} + \cdots \right\}, \tag{29}$$

where the coefficients M_1 and M_2 do not depend on x but are complicated functions of the k_r, ν_r, μ $(r = 1, 2, \ldots, m-1)$,

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k-1} K_{\nu_r}(\lambda_r) \, d\lambda_r \, K_\mu(x\lambda_1 \cdots \lambda_{m-1}) \sim$$

$$2^{(m-1)k - 2m+1} \pi \sum_{\mu, -\mu} \operatorname{cosec}(\mu\pi) \left(2^{2m-4} x^2 \right)^{-\frac{\mu}{2}}$$

$$\times \sum_{r=1}^{m-1} \left\{ \sum_{\nu_r, -\nu_r} \left[\prod_{j=1}^{m-1} \left(\Gamma\left(\frac{-\nu_j - \nu_r}{2} \right) \Gamma\left(\frac{-\nu_j - \nu_r}{2} \right) \right) \right\} \Gamma(-\nu_r) \, \Gamma\left(\frac{k + \nu_r - \mu}{2} \right) \right.$$

$$\left. \left(\frac{e^{\pm i\pi}}{2^{2m-4} x^2} \right)^{\frac{k + \nu_r - \mu}{2}} {}_2F_{2m-3} \left(\begin{array}{c} \frac{k + \nu_r - \mu}{2}, \frac{k + \nu_r + \mu}{2}; \, 2^{4-2m} x^{-2} \\ 1 + \nu_r, 1 + \frac{\nu_r + \nu_1}{2}, \ldots, 1 + \frac{\nu_r + \nu_{m-1}}{2}, \frac{\nu_r - \nu_1}{2} + 1, 1 + \frac{\nu_r - \nu_{m-1}}{2} \end{array} \right) \right]\right\}$$

where, for simplicity, we have taken $k_r = k(r = 1, 2, \ldots, m - 1$, $\qquad$ (30)

To prove (29), substitute from (9) for the E-functions appearing on the right,

then it becomes

$$\sum_{r=1}^{m-1} k_r - 2m+1$$
$$2 \qquad - \frac{1}{2\pi i} \int \frac{\Gamma(\frac{\mu}{2} - s)\,\Gamma(-\frac{\mu}{2} - s)}{1} \prod_{r=1}^{m-1} \{\Gamma(\frac{k_r + \nu_r}{2} - s)$$

$$\Gamma(\frac{k_r - \nu_r}{2} - s)\} \, (\frac{x^2}{2^{2m}})^s \, ds \; .$$

$$= 2^{\sum_{r=1}^{m-1} \cdot_r - 2m+1} \qquad \cdot \, I \; . \qquad\qquad (31)$$

We next show that

$$(2\pi)^{\frac{1}{2} - m} (2m)^{\frac{1}{2}} \exp\{(2m)\,(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}\} I = -\frac{1}{2\pi i} \int S(s)\,(\frac{x^2}{2^{2m}})^s \, ds \; , \quad (32)$$

where

$$S(s) = \sum_{n=0}^{\infty} \Gamma(\frac{\mu}{2} + \frac{u}{2m} - s)\,\Gamma(-\frac{\mu}{2} + \frac{u}{2m} - s)$$

$$\times \frac{\prod\limits_{r=1}^{m-1} \{\Gamma(\frac{k_r + \nu_r}{2} + \frac{u}{2m} - s)\,\Gamma(\frac{k_r - \nu_r}{2} + \frac{u}{2m} - s)\}}{2m \prod\limits_{r=1}^{2m} \{\Gamma(\frac{u+r}{2m})\}} \; . \qquad (33)$$

To see this we substitute the value of $S(s)$ in (32), the right hand side

becomes

$$-\frac{1}{2\pi i}\int_{}^{\infty}\sum_{u=0}\Gamma(\frac{\mu}{2}+\frac{u}{2m}-s)\ \Gamma(-\frac{\mu}{2}+\frac{u}{2m}-s)$$

$$\times\ \frac{\prod\limits_{r=1}^{m-1}\{\Gamma(\frac{k_r+\nu_r}{2}+\frac{u}{2m}-s)\ \Gamma(\frac{k_r-\nu_r}{2}+\frac{u}{2m}-s)\}}{\prod\limits_{r=1}^{2m}\{\Gamma(\frac{u+r}{2m})\}}\ (\frac{x^2}{2^{2m}})^s\ ds\ .$$

Here change $s\to s+\dfrac{u}{2m}$ and obtain

$$-\frac{1}{2\pi i}\sum_{u=0}^{\infty}(\frac{x^2}{2^{2m}})^{\frac{u}{2m}}\ [\ \prod_{r=1}^{2m}\{\Gamma(\frac{u+r}{2m})\}]^{-1}$$

$$\times\int\Gamma(\frac{\mu}{2}-s)\ \Gamma(-\frac{\mu}{2}-s)\prod_{r=1}^{m-1}\{\Gamma(\frac{k_r+\nu_r}{2}-s)\ \Gamma(\frac{k_r-\nu_r}{2}-s)\}(\frac{x^2}{2^{2m}})^s\ ds\ ,$$

which is, in virtue of (31), equal to

$$\sum_{n=0}^{\infty}[(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}]^u\{\Gamma(\frac{u+1}{2m})\ \cdots\ \Gamma(\frac{u+1}{2m}+\frac{2m-1}{2m})\}^{-1}\ I\ .$$

We now use (7) with $2m$ instead of m to obtain

$$\sum_{n=0}^{\infty}[(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}]^u\{\frac{(2\pi)^{m-\frac{1}{2}}(2m)^{\frac{1}{2}-(u+1)}}{\Gamma(u+1)}\}^{-1}\ I\ ,$$

which is equal to

$$(2\pi)^{\frac{1}{2}-m}(2m)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\{(2m)\ (\frac{x^2}{2^{2m}})^{\frac{1}{2m}}\}^u\frac{1}{u!}\ I$$

$$=(2\pi)^{\frac{1}{2}-m}(2m)^{-\frac{1}{2}}\exp\{(2m)(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}\}\ I\ .$$

With (32) established, we utilize the results of Barnes on the asymptotic expansions of the generalized hypergeometric functions (see [7], pp. 296-297, and [8], pp. 80, 108, 110) - for the asymptotic expansion for the expression on

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the right of (32), which holds when x is large and so

$$I \sim \exp\{-2m(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}\}\, (\frac{x^2}{2^{2m}})^{\frac{1}{2m}[\sum_{r=1}^{m-1} k_r + \frac{1}{2} - m]}$$

$$\times \{\frac{(2\pi)^{m-\frac{1}{2}}}{\sqrt{(2m)}} + \frac{M_1}{(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}} + \frac{M_2}{(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}} + \ldots\}$$

and so (29) is established.

By applying (5), formula (30) can be established for large values of x .

Of course the leading term in the asymptotic expansion of (29) is

$$2^{\sum_{r=1}^{k-1} k_r - 2m+1} (2\pi)^{m-\frac{1}{2}} \exp\{-2m(\frac{x^2}{2^{2m}})^{\frac{1}{2m}}\}$$

$$\times (\frac{x^2}{2^{2m}})^{\frac{1}{2m}[\sum_{r=1}^{m-1} k_r + \frac{1}{2} - m]} \qquad . \qquad (34)$$

We now consider the special case $k_r = \frac{2r}{m}$ and $\nu_r = \mu(r = 1, 2, \ldots, m-1)$ and show that the result (34) agrees with the asymptotic expansion of $K_{m\mu}(mx^{\frac{1}{m}})$ which appears on the right of (21). We substitute $k_r = \frac{2r}{m}$, $\nu_r = \mu\,(r = 1, 2, \ldots, m-1)$ in (34) and it becomes

$$2^{m-1-2m+1} (2\pi)^{m-\frac{1}{2}} (2m)^{-\frac{1}{2}} \exp\left\{-2m\left(\frac{x^2}{2^{2m}}\right)^{\frac{1}{2m}}\right\}$$

$$\left(\frac{x^2}{2^{2m}}\right)^{\frac{1}{2m}(m-1+\frac{1}{2}-m)}$$

$$= 2^{m-m-\frac{1}{2}} m^{-\frac{1}{2}} (\pi)^{m-\frac{1}{2}} \exp\left\{-2m\left(\frac{x^2}{2^{2m}}\right)^{\frac{1}{2m}}\right\} x^{-\frac{1}{2m}}$$

$$= \pi^{m-1} \frac{1}{\sqrt{(2\pi x^{1/m})}} e^{-m\, x^{1/m}} ,$$

which agrees with the asumptotic expansion of the right hand side of (21)

because

$$K_\mu(x) \sim \frac{1}{\sqrt{(2\pi x)}} e^{-x} \quad \text{as} \quad |x| \to \infty .$$

Again we can write the asymptotic expansions of the integrals (1) and (2) as $|x| \to \infty$. For the asymptotic expansion of (1) as $|x| \to \infty$ is obtained by expanding each E-function on the right by means of (5), and combining the two resulting expressions by factoring out common terms.

The asymptotic expansion of the integral (2) can be written at once in virtue of (28) in the form

$$\prod_{r=1}^{m-1} \int_0^\infty \lambda_r^{k_r-1} K_{\nu_r}(\lambda_r)\, d\lambda_r\, K_\mu(x\lambda_1\lambda_2\cdots\lambda_{m-1}) \sim$$

$$2^{\sum\limits_{r=1}^{m-1} k_r - 2m+1} \cdot \pi \sum_{\mu,-\mu}\left[\operatorname{cosec}\mu\pi\,(2^{2m-2}x^2)^{-\frac{\mu}{2}}\right.$$

$$\frac{\prod\limits_{r=1}^{m-1}\left\{\Gamma(\dfrac{k_r+\nu_r-\mu}{2})\ \Gamma(\dfrac{k_r-\nu_r-\mu}{2})\right\}}{\Gamma(1-\mu)}$$

$$\left. {}_{2m-2}F_1\left(\frac{k_1-\nu_1-\mu}{2}\,,\ \frac{k_1+\nu_1-\mu}{2}\,,\ \cdots\,,\ \frac{k_{m-1}-\nu_{m-1}-\mu}{2}\,,\ \frac{k_{m-1}+\nu_{m-1}-\mu}{2}\,;\ 2^{2m-4}x^2 \atop 1-\mu \right)\right].$$

(35)

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